International Journal of Theoretical Physics, Vol. 37, No. 2, 1998

Gravitational and Electroweak Interactions

Dave Pandres, Jr.[1]

Received September 20, 1997

Schrödinger considered the variational principle $\delta \int \sqrt{-g}\, d^4x = 0$, where g is the determinant of the metric $g_{\mu\nu}$, but noted that if $g_{\mu\nu}$ is varied, the resulting Euler–Lagrange equations cannot serve as field equations. We write $g_{\mu\nu} = g_{ij} h^i{}_\mu h^j{}_\nu$, where $g_{ij} = \mathrm{diag}(-1, 1, 1, 1)$, and express the vectors of the tetrad $h^i{}_\mu$ as derivatives of nonintegrable functions x^i of the type commonly used for phase factors in gauge theory, i.e., $h^i{}_\mu = x^i{}_{,\mu}$. We have previously shown that if the x^i are varied, the resulting Euler–Lagrange equations serve as field equations which imply the validity of Einstein equations with a stress-energy tensor for the electroweak field and associated currents. In this paper, we express these Einstein equations into two new forms, and use these forms to derive Lorentz-force-like equations of motion. The electroweak field appears as a consequence of the field equations (rather than as a "compensating field" introduced to secure local gauge invariance). There is no need for symmetry breaking to accommodate mass, because the gauge symmetry is approximate from the outset.

1. INTRODUCTION

Schrödinger (1960) recognized that the simplest general relativistic variational principle which exists is

$$\delta \int \sqrt{-g}\, d^4x = 0 \tag{1}$$

where g is the determinant of the metric $g_{\mu\nu}$. He noted, however, that variation of $g_{\mu\nu}$ yields the Euler–Lagrange equations $\sqrt{-g}\; g^{\mu\nu} = 0$, which cannot serve as field equations. Lowercase indices take the values 0, 1, 2, 3, and the summation convention is adopted. Any space-time metric may be expressed in terms of a tetrad of vectors $h^i{}_\mu$, i.e., $g_{\mu\nu} = g_{ij} h^i{}_\mu h^j{}_\nu$, where $g_{ij} = g^{ij} = \mathrm{diag}(-1, 1, 1, 1)$. We use g^{ij} and g_{ij} to raise and lower tetrad (Latin) indices,

[1] Department of Mathematics and Computer Science, North Georgia College and State University, Dahlonega, Georgia 30597.

0020-7748/98/0300-82 7$15.00/0

just as we use $g^{\mu\nu}$ and $g_{\mu\nu}$ to raise and lower space-time (Greek) indices, e.g., $h_{i\mu} = g_{ij}h^{j}{}_{\mu}$.

We now recall (Pandres, 1995) the step that is crucial for the development of our theory: *We express the vectors of the tetrad as derivatives of "nonintegrable functions"* x^i, and vary the x^i. [A nonintegrable function does not have a definite numerical value at a point, but its derivatives have definite values at a point. Such nonintegrable (path-dependent) functions have been used as phase factors by Dirac (1978), Yang (1974), and many others in gauge theory.] Any tetrad may be expressed in this way; thus, without loss of generality, we may write $h^{i}{}_{\mu} = x^{i}{}_{,\mu}$, where a comma denotes partial differentiation.

The field equations that flow from equation (1) are (Pandres, 1995)

$$C_\mu = 0 \tag{2}$$

where the vector C_μ is defined by

$$C_\mu = h_i^{\nu}(h^{i}{}_{\mu,\nu} - h^{i}{}_{\nu,\mu}) \tag{3}$$

(*Note:* The quantity $h^{i}{}_{\mu,\nu} - h^{i}{}_{\nu,\mu}$ equals $x^{i}{}_{,\mu,\nu} - x^{i}{}_{,\nu,\mu}$, which would vanish if the x^i were ordinary functions; thus, our field equations would reduce to trivial identities.) We show in Section 2 that these field equations imply the validity of Einstein's equations for general relativity with a stress-energy tensor which is just what one expects for a non-Abelian gauge field and its associated currents. Evidence is presented that this gauge field is the electroweak field. Roughly speaking, transformations involving only the tetrad index 0 correspond to $U(1)$, while those that mix the tetrad indices 1, 2, 3 correspond to $SU(2)$. The gauge field appears in the theory as a direct consequence of the field equations, rather than being introduced as a "compensating field" to secure local gauge invariance (as in the standard development of gauge theory). There is no need for symmetry breaking to accommodate mass, because the gauge symmetry is approximate from the outset. We include a discussion of the special case in which only gravitation and electromagnetism are present. In Section 3, we derive Lorentz-force-like equations of motion from the Einstein equations. In Section 4, we discuss an alternative theory based on a different variational principle that yields different field equations, which, however, imply the validity of Einstein equations that are identical in form to those of Section 2, but slightly different in interpretation.

We have shown previously (Pandres, 1981) that

$$h^{i}{}_{\mu} = \delta^{i}_{\mu} + \delta^{i}_{0}\delta^{2}_{\mu}\delta^{1}_{\alpha}x^{\alpha} \tag{4}$$

is a solution to our field equations, and that this solution yields the well-known (Synge, 1960) Einstein equations for an electrically charged dust cloud. Synge showed that the equations of motion which are derived from these Einstein equations just state that electromagnetic current flows according to the Lorentz force law.

The first suggestion that our theory describes the electroweak field as well as the gravitational field can already be seen at this early stage of the analysis. Equation (3) can obviously be written

$$C_\mu = -h_i^{\ \nu} f^i_{\ \mu\nu} \tag{5}$$

where $f^i_{\ \mu\nu}$ is the curl of $h^i_{\ \mu}$, i.e.,

$$f^i_{\ \mu\nu} = h^i_{\ \nu,\mu} - h^i_{\ \mu,\nu} \tag{6}$$

We define

$$\mathfrak{F}^i_{\ \mu\nu} = h^i_{\ \nu,\mu} - h^i_{\ \mu,\nu} + e^{0ijk} h_{j\mu} h_{k\nu} \tag{7}$$

where e^{nijk} is the usual Levi-Civita symbol. We note that $\mathfrak{F}^i_{\ \mu\nu}$ is the usual field strength for a $U(1) \times SU(2)$ gauge field, *provided that $h^i_{\ \mu}$ is transformed on its tetrad indices as a gauge potential,* rather than as a Lorentz vector. By using the antisymmetry of e^{0ijk} in i and k, we easily find that $h_i^{\ \nu} e^{0ijk} h_{j\mu} h_{k\nu} = 0$. From this and equation (7), we see that equation (5) may be written

$$C_\mu = -h_i^{\ \nu} \mathfrak{F}^i_{\ \mu\nu} \tag{8}$$

Thus, as we have noted previously (Pandres, 1995), *the form of our field equations is unchanged when the curl $f^i_{\ \mu\nu}$ is replaced by the gauge field $\mathfrak{F}^i_{\ \mu\nu}$*. It is clear that if $h^i_{\ \mu}$ is transformed on its tetrad indices as a gauge potential, then the metric $g_{\mu\nu} = g_{ij} h^i_{\ \mu} h^j_{\ \nu}$ is generally changed. It is eminently reasonable that when a particle is subjected to a gauge transformation which changes its mass, the gravitational field also should change.

1.1. Ricci Rotation Coefficients

We follow Eisenhart (1925) in defining Ricci rotation coefficients by $\gamma_{i\mu\nu} = h_{i\mu;\nu}$, where a semicolon denotes the usual covariant derivative of Riemannian geometry. The relation $\gamma_{\mu\nu i} = h^j_{\ \mu} \gamma_{j\nu\alpha} h_i^{\ \alpha}$ illustrates our general method for converting between Greek and Latin indices. It is well known that $\gamma_{\mu\nu i}$ is antisymmetric in μ and ν. This antisymmetry may be used to obtain an expression for $\gamma_{\mu\nu i}$ in terms of $f_{i\mu\nu}$. We have $f_{i\mu\nu} = h_{i\nu,\mu} - h_{i\mu,\nu} = h_{i\nu;\mu} - h_{i\mu;\nu}$, so that $f_{i\mu\nu} = \gamma_{i\nu\mu} - \gamma_{i\mu\nu}$. If we subtract from this the corresponding expressions for $f_{\mu\nu i}$ and $f_{\nu i\mu}$, we obtain

$$\gamma_{\mu\nu i} = \tfrac{1}{2}(f_{i\mu\nu} - f_{\mu\nu i} - f_{\nu i\mu}) \tag{9}$$

From equation (9), we find that $\gamma^\nu_{\ \mu\nu} = f^\nu_{\ \nu\mu} = h_i^{\ \nu}(h^i_{\ \mu,\nu} - h^i_{\ \nu,\mu})$, i.e., that

$$C_\mu = \gamma^\nu{}_{\mu\nu} \tag{10}$$

2. EINSTEIN EQUATIONS

The Einstein equations of general relativity may be interpreted in two ways. One interpretation is as differential equations for the metric, when the stress-energy tensor is given. Alternatively, these equations may be looked upon as a definition of the stress-energy tensor in terms of the metric. The second interpretation has been stressed particularly by Schrödinger (1960) ("I would rather you did not regard these equations as field equations, but as a definition of T_{ik} the matter tensor") and by Eddington (1957) ("and we must proceed by inquiring first what experimental properties the physical tensor possesses, and then seeking a geometrical tensor which possesses these properties"). It is the second interpretation that we adopt.

We define the Riemann curvature tensor in the usual way

$$R^\alpha{}_{\beta\mu\nu} = h_i{}^\alpha(h^i{}_{\beta;\mu;\nu} - h^i{}_{\beta;\nu;\mu}) \tag{11}$$

and define the Ricci tensor $R_{\mu\nu}$, Ricci scalar R, and Einstein tensor $G_{\mu\nu}$ as usual by $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$, $R = R^\alpha{}_\alpha$, and $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, respectively. By using

$$h_i{}^\alpha h^i{}_{\beta;\mu;\nu} = (h_i{}^\alpha h^i{}_{\beta;\mu})_{;\nu} - h_i{}^\alpha{}_{;\nu}h^i{}_{\beta;\mu} = \gamma^\alpha{}_{\beta\mu;\nu} + \gamma^\alpha{}_{\sigma\nu}\gamma^\sigma{}_{\beta\mu}$$

we find from equation (11) that

$$R^\alpha{}_{\beta\mu\nu} = \gamma^\alpha{}_{\beta\mu;\nu} - \gamma^\alpha{}_{\beta\nu;\mu} + \gamma^\alpha{}_{\sigma\nu}\gamma^\sigma{}_{\beta\mu} - \gamma^\alpha{}_{\sigma\mu}\gamma^\sigma{}_{\beta\nu} \tag{12}$$

which is the Ricci identity for the Riemann tensor. From equation (12), we obtain the following identity for the Ricci tensor:

$$R_{\mu\nu} = C_{\mu;\nu} - C_\alpha\gamma^\alpha{}_{\mu\nu} - \gamma^\alpha{}_{\mu\nu;\alpha} + \gamma^\alpha{}_{\sigma\nu}\gamma^\sigma{}_{\mu\alpha} \tag{13}$$

Our field equations just state that $C_\mu = 0$; thus, they imply that

$$R_{\mu\nu} = -\gamma^\alpha{}_{\mu\nu;\alpha} + \gamma^\alpha{}_{\sigma\nu}\gamma^\sigma{}_{\mu\alpha} \tag{14}$$

Now $\gamma^\alpha{}_{\mu\nu;\alpha}$ may be expressed in terms of the conserved current $j^i{}_\mu = f^i{}_\mu{}^\alpha{}_{;\alpha}$ which is the source of the field $f^i{}_{\mu\nu}$. From equation (9), we easily obtain the expression $-\gamma^\alpha{}_{\mu\nu} = \frac{1}{2}(f_{\mu\nu}{}^\alpha + f_{\nu\mu}{}^\alpha + f^\alpha{}_{\mu\nu})$. By using this, we write equation (14) as

$$R_{\mu\nu} = \tfrac{1}{2}(f_{\mu\nu}{}^\alpha{}_{;\alpha} + f_{\nu\mu}{}^\alpha{}_{;\alpha} + f^\alpha{}_{\mu\nu;\alpha}) + \gamma^\alpha{}_{\sigma\nu}\gamma^\sigma{}_{\mu\alpha} \tag{15}$$

Notice that

$$f_{\mu\nu}{}^\alpha{}_{;\alpha} = (h^i{}_\mu f_{i\nu}{}^\alpha)_{;\alpha} = h^i{}_{\mu;\alpha}f_{i\nu}{}^\alpha + h^i{}_\mu f_{i\nu}{}^\alpha{}_{;\alpha} = \gamma^i{}_{\mu\alpha}f_{i\nu}{}^\alpha + h^i{}_\mu j_{i\nu}$$

Also,

$$\gamma^{i}{}_{\mu\alpha}f_{i\nu}{}^{\alpha} = \gamma^{i}{}_{\mu}{}^{\alpha}f_{i\nu\alpha} = \gamma^{i}{}_{\mu}{}^{\alpha}(\gamma_{i\alpha\nu} - \gamma_{i\nu\alpha}) = \gamma_{i\alpha\nu}\gamma^{i}{}_{\mu}{}^{\alpha} - \gamma^{i}{}_{\mu}{}^{\alpha}\gamma_{i\nu\alpha}$$
$$= \gamma^{\sigma}{}_{\mu}{}^{\alpha}\gamma_{\sigma\alpha\nu} - \gamma^{\alpha}{}_{\mu}{}^{i}\gamma_{\alpha\nu i} = -\gamma^{\sigma}{}_{\mu}{}^{\alpha}\gamma_{\alpha\sigma\nu} - \gamma^{\alpha}{}_{\mu i}\gamma_{\alpha\nu}{}^{i}$$
$$= -\gamma^{\alpha}{}_{\sigma\nu}\gamma^{\sigma}{}_{\mu\alpha} - \gamma^{\alpha}{}_{\mu i}\gamma_{\alpha\nu}{}^{i}$$

By using these expressions for $f_{\mu\nu}{}^{\alpha}{}_{;\alpha}$ and $\gamma^{i}{}_{\mu\alpha}f_{i\nu}{}^{\alpha}$ in equation (15), we have

$$R_{\mu\nu} = \tfrac{1}{2}(h^{i}{}_{\mu}j_{i\nu} + h^{i}{}_{\nu}j_{i\mu}) - \gamma^{\alpha}{}_{\mu i}\gamma_{\alpha\nu}{}^{i} + \tfrac{1}{2}(f^{\alpha}{}_{\mu\nu;\alpha} + \gamma^{\alpha}{}_{\sigma\nu}\gamma^{\sigma}{}_{\mu\alpha} - \gamma^{\alpha}{}_{\sigma\mu}\gamma^{\sigma}{}_{\nu\alpha}) \qquad (16)$$

We now use the symmetry of $R_{\mu\nu}$ to rewrite equation (16) as

$$R_{\mu\nu} = \tfrac{1}{2}(h^{i}{}_{\mu}j_{i\nu} + h^{i}{}_{\nu}j_{i\mu}) - \gamma^{\alpha}{}_{\mu i}\gamma_{\alpha\nu}{}^{i} \qquad (17)$$

We find from equation (17) that

$$R = h^{i\alpha}j_{i\alpha} - \gamma^{\alpha\beta i}\gamma_{\alpha\beta i} \qquad (18)$$

2.1. Einstein Equations (First Form)

From equations (17) and (18), we obtain the Einstein equations

$$G_{\mu\nu} = \tfrac{1}{2}(h^{i}{}_{\mu}j_{i\nu} + h^{i}{}_{\nu}j_{i\mu} - g_{\mu\nu}h^{i\alpha}j_{i\alpha}) - \gamma_{\mu\nu} + \tfrac{1}{4}g_{\mu\nu}\gamma^{\alpha\beta i}\gamma_{\alpha\beta i} \qquad (19)$$

where

$$\gamma_{\mu\nu} = \gamma^{\alpha}{}_{\mu i}\gamma_{\alpha\nu}{}^{i} - \tfrac{1}{4}g_{\mu\nu}\gamma^{\alpha\beta i}\gamma_{\alpha\beta i} \qquad (20)$$

has the formal structure of a stress-energy tensor for a non-Abelian gauge field. In equation (20), the formal role of the gauge field strength is played by the Ricci rotation coefficient $\gamma_{\mu\nu i}$.

It is tempting to interpret the last term in equation (19) as a "cosmological" term that might account for an expanding universe, or for the mysterious "dark matter." Equation (18) shows that in a source-free region this last term is just $-\frac{1}{4}g_{\mu\nu}R$.

The spin connection in general relativity is $\Gamma_{\mu} = \frac{1}{8}\gamma_{ij\mu}(\gamma^{i}\gamma^{j} - \gamma^{j}\gamma^{i}) + a_{\mu}I$, where the γ^{i} are the Dirac matrices of special relativity, I is the identity matrix, and a_{μ} is an arbitrary vector. (A Lorentz transformation on tetrad indices corresponds to a similarity transformation of the spinors.) Now, it is well known that the spin connection contains complete information about the electromagnetic field, and that one half of Maxwell's equations are identically satisfied on account of the existence of the spin connection. Furthermore, the manner in which the electromagnetic field enters the spin connection is in agreement with the principle of minimal electromagnetic coupling. An understanding of the spinor calculus in Riemann space, and the role played by the spin connection, was gained through the work of many investigators during the decade after Dirac's discovery of the relativistic theory of the electron; see, e.g., Bade and Jehle (1953) for a general review. Many of these investigators recognized the description of the electromagnetic field as part of the spin connection. An especially lucid discussion of this was given by Loos (1963). The subsequent unification of the electromagnetic and weak fields by Weinberg (1967) and Salam (1968) leads us to expect that the spin connection might also contain a description of the weak field. In Section 3.1 we shall present evidence that the electroweak field is described by the "mixed symmetry" part of $\gamma_{\mu\nu i}$ under the permutation group on three symbols. First, however, we consider the special case in which only gravitation and electromagnetism are present.

2.2. Einstein Equations for Gravitation and Electromagnetism

We recall (Pandres, 1981) that if $f^i{}_{\mu\nu} = 0$ for $i = 1, 2, 3$, then

$$G_{\mu\nu} = -\tfrac{1}{2}(h^0{}_\mu j^0{}_\nu + h^0{}_\nu j^0{}_\mu) + \tfrac{1}{2}(f^{0\alpha}{}_\mu f^0{}_{\alpha\nu} - \tfrac{1}{4}g_{\mu\nu}f^{0\alpha\beta}f^0{}_{\alpha\beta}) - Rh^0{}_\mu h^0{}_\nu \tag{21}$$

The orthodox physical interpretation, which we adopt, is that $h^i{}_\mu$ describes an observer frame. Thus $h^0{}_\mu$, which is the vector potential for $f^0{}_{\mu\nu}$, is also the (timelike) velocity vector of an observer carrying a spatial frame described by the triad $h^I{}_\mu$, where capital Latin indices take the values 1, 2, 3. We denote $h^0{}_\mu = v_\mu$, $f^0{}_{\mu\nu} = f_{\mu\nu}$, and $j^0{}_\mu = j_\mu$, so that equation (21) becomes

$$G_{\mu\nu} = -\tfrac{1}{2}(v_\mu j_\nu + v_\nu j_\mu) + \tfrac{1}{2}E_{\mu\nu} - Rv_\mu v_\nu \tag{22}$$

where

$$E_{\mu\nu} = f^\alpha{}_\mu f_{\alpha\nu} - \tfrac{1}{4}g_{\mu\nu}f^{\alpha\beta}f_{\alpha\beta} \tag{23}$$

is the usual stress-energy tensor for the electromagnetic field, and j_μ is the usual electromagnetic current.

2.3. The Electroweak Field

We now recall evidence (Pandres, 1995) that the electroweak field is described by the mixed symmetry part of $\gamma_{\mu\nu i}$ under the permutation group on three symbols. This group has six group elements. One group element is the identity. The other five group elements are "cycles" such as $(\mu\nu i)$, which has the effect of replacing μ with ν, ν with i, and i with μ. These five group elements are $(\mu\nu)$, (νi), $(i\mu)$, $(\mu i\nu)$, and $(\mu\nu i)$. The Ricci rotation coefficients may be decomposed into their totally antisymmetric and mixed symmetry parts. (The totally symmetric part vanishes because the coefficients are antisymmetric in their first two indices.) The totally antisymmetric part of $\gamma_{\mu\nu i}$ is

$$A_{\mu\nu i} = \frac{1}{3}(\gamma_{\mu\nu i} + \gamma_{i\mu\nu} + \gamma_{\nu i\mu}) \tag{24}$$

The mixed symmetry part of $\gamma_{\mu\nu i}$ is the quantity

$$M_{\mu\nu i} = \gamma_{\mu\nu i} - A_{\mu\nu i} = \frac{1}{3}(2\gamma_{\mu\nu i} - \gamma_{i\mu\nu} - \gamma_{\nu i\mu}) \tag{25}$$

which is antisymmetric in μ and ν. Thus, we have

$$\gamma_{\mu\nu i} = A_{\mu\nu i} + M_{\mu\nu i} \tag{26}$$

and we see from equations (10) and (26) that

$$C_\mu = M^\mu{}_{\mu\nu} \tag{27}$$

It follows from equation (25) that

$$M_{\mu\nu i} + M_{i\mu\nu} + M_{\nu i\mu} = 0 \tag{28}$$

We note that $M_{\mu\nu i}$ may be expressed in terms of $f_{i\mu\nu}$. From equations (25) and (9) we find that

$$M_{\mu\nu i} = \frac{1}{3}(2f_{i\mu\nu} - f_{\mu\nu i} - f_{\nu i\mu}) = \frac{2}{3}\gamma_{\mu\nu i} + \frac{1}{3}f_{i\mu\nu} \tag{29}$$

and this may be written

$$M_{\mu\nu i} = \frac{1}{3}(2\delta_i^n\delta_\mu^\alpha\delta_\nu^\sigma - h^n{}_\mu\delta_\nu^\alpha h_i^\sigma - h^n{}_\nu h_i{}^\alpha\delta_\mu^\sigma)f_{n\alpha\sigma} \tag{30}$$

Upon using equations (6) and (7) in equation (30), we obtain

$$M_{\mu\nu i} = \frac{1}{3}(2\delta_i^n\delta_\mu^\alpha\delta_\nu^\sigma - h^n{}_\mu\delta_\nu^\alpha h_i^\sigma - h^n{}_\nu h_i{}^\alpha\delta_\mu^\sigma)(\mathfrak{F}_{n\alpha\sigma} - e^0{}_n{}^{jk}h_{j\alpha}h_{k\sigma}) \tag{31}$$

It is easily verified that

$$(2\delta_i^n\delta_\mu^\alpha\delta_\nu^\sigma - h^n{}_\mu\delta_\nu^\alpha h_i{}^\sigma - h^n{}_\nu h_i{}^\alpha\delta_\mu^\sigma)e^0{}_n{}^{jk}h_{j\alpha}h_{k\sigma} = 0$$

Therefore, equation (31) reduces to

$$M_{\mu\nu i} = \frac{1}{3}(2\delta_i^n\delta_\mu^\alpha\delta_\nu^\sigma - h^n{}_\mu\delta_\nu^\alpha h_i^\sigma - h^n{}_\nu h_i^\alpha\delta_\mu^\sigma)\mathfrak{F}_{n\alpha\sigma} \tag{32}$$

From equation (32), we see that in the expression (30) for $M_{\mu\nu i}$ the curl $f_{i\mu\nu}$ may simply be replaced by the gauge field $\mathfrak{F}_{i\mu\nu}$ [just as in the expression (5) for C_μ]. We shall see that the quantity $\mathfrak{F}_{i\mu\nu}$ does not directly describe the electroweak field. It is, however, the fundamental ingredient which is essential for the description of that field. Indeed, $\mathfrak{F}_{n\alpha\sigma}$ in (32) may be viewed as a field with "bare" or massless quanta, which are "clothed" by the factor $\frac{1}{3}(2\delta_i^n\delta_\mu^\alpha\delta_\nu^\sigma - h^n{}_\mu\delta_\nu^\alpha h_i^\sigma - h^n{}_\nu h_i{}^\alpha\delta_\mu^\sigma)$, and thus may acquire mass. It is $M_{\mu\nu i}$ that we tentatively identify as the physical electroweak field.

2.3.1. Affine Connection for Quantities with Tetrad Indices

We shall see that it is useful to regard the negative of $A^i_{j\nu}$ as an affine connection for "total" covariant differentiation of quantities with tetrad indices. We use a stroke | to denote the total covariant derivative. Thus, for the total covariant derivatives of $h^i{}_\mu$ and $h_{i\mu}$, we have

$$h^i{}_{\mu|\nu} = h^i{}_{\mu;\nu} - h^j{}_\mu A^i{}_{j\nu} = M^i{}_{\mu\nu} \tag{33}$$

$$h_{i\mu|\nu} = h_{i\mu;\nu} + h_{j\mu} A^j{}_{i\nu} = M_{i\mu\nu}$$

For a quantity that has only space-time indices, there is no distinction between "ordinary" and "total" covariant differentiation.

It is important to observe from (28) and (33) that $M_{\mu\nu i}$ is just the total curl of $h_{i\mu}$, i.e.,

$$M_{\mu\nu i} = h_{i\nu|\mu} - h_{i\mu|\nu} \tag{34}$$

This observation gives us further encouragement to identify $M_{\mu\nu i}$ tentatively as the electroweak field. For this identification to be valid, the quantity

$$M_{\mu\nu 0} = \tfrac{1}{3}(2f_{0\mu\nu} - f_{\mu\nu 0} - f_{\nu 0\mu}) \tag{35}$$

must describe the electromagnetic field. The presence of the extra terms $-f_{\mu\nu 0} - f_{\nu 0\mu}$ in equation (35) may lead one to ask how $M_{\mu\nu i}$ can be identified as the electroweak field. Our answer is this: *If $h^i{}_\mu$ describes a freely falling, nonrotating observer frame, then equation (35) reduces to $M_{\mu\nu 0} = \frac{1}{3}f_{0\mu\nu}$.* This may be seen as follows. The vector field $h^0{}_\mu$ is tangent to, and therefore defines, a timelike congruence of curves. These are the world lines of an observer with velocity $h^0{}_\mu$ carrying a spatial frame described by $h^I{}_\mu$. To obtain an $h^i{}_\mu$ that describes a freely falling, nonrotating frame, we choose $h^0{}_\mu$ tangent to a timelike geodesic congruence, and carry $h^I{}_\mu$ along the geodesics by parallel transport [to which Fermi–Walker transport reduces (Synge, 1960) along nonnull geodesics]. Thus, the condition for freely falling, nonrotating frames is $h_{i\nu;\alpha} h_0{}^\alpha = 0$. In terms of the Ricci rotation coefficients, the condition is $\gamma_{\mu\nu 0} = 0$. From this and equation (25), we see that for an $h^i{}_\mu$ which describes a freely falling, nonrotating observer frame,

$$M_{\mu\nu 0} = \tfrac{1}{3}(\gamma_{0\nu\mu} - \gamma_{0\mu\nu}) = \tfrac{1}{3}(h_{0\nu;\mu} - h_{0\mu;\nu}) = \tfrac{1}{3}(h_{0\nu,\mu} - h_{0\mu,\nu}) = \tfrac{1}{3}f_{0\mu\nu}$$

2.3.2. Total Einstein Equations

We now recall (Pandres, 1995) more compelling evidence that $M_{\mu\nu i}$ describes the electroweak field. We define a total Riemann tensor

$$\mathfrak{R}^\alpha{}_{\beta\mu\nu} = h_i{}^\alpha(h^i{}_{\beta|\mu|\nu} - h^i{}_{\beta|\nu|\mu}) \tag{36}$$

which is the total analogue of the usual Riemann tensor $R^\alpha{}_{\beta\mu\nu}$. We define a total Ricci tensor by $\mathfrak{R}_{\mu\nu} = \mathfrak{R}^\alpha{}_{\mu\alpha\nu}$, a total Ricci scalar by $\mathfrak{R} = \mathfrak{R}^\alpha{}_\alpha$, and a total Einstein tensor by $\mathfrak{G}_{\mu\nu} = \mathfrak{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathfrak{R}$. We have shown (Pandres, 1995) that an identity for the total Einstein tensor is

$$\mathfrak{G}_{\mu\nu} = C_{\mu|\nu} - C_\alpha M^\alpha{}_{\mu\nu} - g_{\mu\nu} C^\alpha{}_{|\alpha} - \tfrac{1}{2} g_{\mu\nu} C^\alpha C_\alpha + \mathfrak{I}_{\mu i} h^i{}_\nu - (M^\alpha{}_{\mu i} M_{\alpha\nu}{}^i - \tfrac{1}{4} g_{\mu\nu} M^{\alpha\beta i} M_{\alpha\beta i}) \tag{37}$$

where $\mathfrak{I}_{\mu i} = M_\mu{}^\alpha{}_{i|\alpha}$ is the total gauge current. It is not generally conserved. We see from (37) that our field equations $C_\mu = 0$ imply the validity of the total Einstein equations

$$\mathfrak{G}_{\mu\nu} = \mathfrak{I}_{\mu i} h^i{}_\nu - (M^\alpha{}_{\mu i} M_{\alpha\nu}{}^i - \tfrac{1}{4} g_{\mu\nu} M^{\alpha\beta i} M_{\alpha\beta i}) \tag{38}$$

By contrast with the conventional Einstein tensor $G_{\mu\nu}$, our total Einstein tensor $\mathfrak{G}_{\mu\nu}$ is nonsymmetric. We denote its symmetric part by $\mathfrak{G}_{\underline{\mu\nu}}$. The symmetric part of equation (38) is

$$\mathfrak{G}_{\underline{\mu\nu}} = \tfrac{1}{2}(\mathfrak{I}_{\mu i} h^i{}_\nu + \mathfrak{I}_{\nu i} h^i{}_\mu) - (M^\alpha{}_{\mu i} M_{\alpha\nu}{}^i - \tfrac{1}{4} g_{\mu\nu} M^{\alpha\beta i} M_{\alpha\beta i}) \tag{39}$$

The right side of equation (39) is just what one expects for the stress-energy tensor of a non-Abelian gauge field and associated currents.

One may ask whether tetrad indices can be associated with intrinsic spin and also with weak isotopic spin. The possibility of transforming $h^i{}_\mu$ either as a Lorentz vector or as a gauge potential suggests that the answer is "yes." However, if this answer should turn out to be untenable, then (in a worse-case scenario) we could retain the results given here, but use a space-time with dimension greater than four.

Our theory provides a complete unification of the gravitational, electromagnetic, and weak fields. By contrast, it is widely recognized that the electroweak theory of Weinberg and Salam provides only a partial unification of the electromagnetic and weak fields. Crease and Mann (1986) note that "The electroweak theory, as it is called today, does not fully unify the two forces. Nevertheless, it ties them together so firmly that most scientists refer to it as unified." Moriyasu (1983) notes that Weinberg and Salam "began with a product of disconnected groups $U(1)$ and $SU(2)$, and ended up by unifying them through a mixing of the corresponding gauge fields. The reason for the mixing, of course, has nothing to do with gauge theory *per se*. It was built in 'by hand' through the identification of the leptons as the appropriate doublets and singlets of weak isotopic-spin." Like Weinberg–Salam theory, our theory in its present form provides no fundamental explanation for the left–right asymmetry of the weak interactions. However, we mention a possible avenue to such an explanation. In five prior papers (Pandres, 1962, 1981, 1984a,b, 1995) we have pursued Einstein's suggestion that the diffeomorphisms be extended to a larger group. This has led to a geometry based not upon a manifold, but on a space in which *paths*, rather than points are the primary elements. In this geometry, x^i and x^μ are on the same footing, but partial derivatives of $h^i{}_\mu$ with respect to x^α and x^β do not generally commute.

This lack of commutativity corresponds to something like a "vorticity" of space-time that might provide a fundamental explanation for the left–right asymmetry of the weak interactions. It also appears possible that the path space could provide a fundamental foundation for string theory. The need for such a foundation has been emphasized especially by Witten (1988).

Equation (39) is not convenient for the purpose of deriving equations of motion, because $\mathfrak{G}_{\mu\nu}{}^{;\mu}$ does not vanish. There is, however, another form of the Einstein equations which, like equation (19), is convenient for this purpose.

2.4. Einstein Equations (Second Form)

We note that

$$\begin{aligned}\gamma^{\alpha}{}_{\mu\nu;\alpha} &= (\gamma^{\alpha}{}_{\mu i}h^{i}{}_{\nu})_{;\alpha} = \gamma^{\alpha}{}_{\mu i;\alpha}h^{i}{}_{\nu} + \gamma^{\alpha}{}_{\mu i}h^{i}{}_{\nu;\alpha}\\ &= -\gamma_{\mu}{}^{\alpha}{}_{i;\alpha}h^{i}{}_{\nu} + \gamma^{\alpha}{}_{\mu i}\gamma^{i}{}_{\nu\alpha} = -J_{\mu i}h^{i}{}_{\nu} + \gamma^{\sigma}{}_{\mu\alpha}\gamma^{\alpha}{}_{\nu\sigma}\end{aligned}$$

where $J_{\mu i} = \gamma_{\mu}{}^{\alpha}{}_{i;\alpha}$ is the conserved current which the source of the field $\gamma_{\mu\nu i}$. Using this expression for $\gamma^{\alpha}{}_{\mu\nu;\alpha}$, and the relation $\gamma^{\alpha}{}_{\sigma\nu} - \gamma^{\alpha}{}_{\nu\sigma} = f^{\alpha}{}_{\nu\sigma}$, in equation (14) gives

$$R_{\mu\nu} = J_{\mu i}h^{i}{}_{\nu} - \gamma^{\alpha}{}_{\mu i}f^{i}{}_{\alpha\nu} \tag{40}$$

Upon multiplying equation (14) by $g^{\mu\nu}$, we obtain

$$\begin{aligned}R &= -\gamma^{\alpha\beta}{}_{\beta;\alpha} + \gamma^{\sigma\beta\alpha}\gamma_{\alpha\sigma\beta} = C^{\alpha}{}_{;\alpha} + \gamma^{\sigma\beta\alpha}\gamma_{\alpha\sigma\beta}\\ &= \tfrac{1}{2}(\gamma^{\sigma\beta\alpha}\gamma_{\alpha\sigma\beta} + \gamma^{\sigma\beta\alpha}\gamma_{\alpha\sigma\beta}) = \tfrac{1}{2}(\gamma^{\sigma\beta\alpha}\gamma_{\alpha\sigma\beta} + \gamma^{\beta\sigma\alpha}\gamma_{\alpha\beta\sigma})\\ &= \tfrac{1}{2}(\gamma^{\sigma\beta\alpha}\gamma_{\alpha\sigma\beta} - \gamma^{\sigma\beta\alpha}\gamma_{\alpha\beta\sigma}) = \tfrac{1}{2}\gamma^{\sigma\beta\alpha}(\gamma_{\alpha\sigma\beta} - \gamma_{\alpha\beta\sigma})\\ &= \tfrac{1}{2}\gamma^{\sigma\beta\alpha}f_{\alpha\beta\sigma} = -\tfrac{1}{2}\gamma^{\beta\sigma\alpha}f_{\alpha\beta\sigma} = -\tfrac{1}{2}\gamma^{\alpha\beta i}f_{i\alpha\beta}\end{aligned}$$

From this expression for R and equation (40), we have

$$G_{\mu\nu} = J_{\mu i}h^{i}{}_{\nu} - \gamma^{\alpha}{}_{\mu i}f^{i}{}_{\alpha\nu} + \tfrac{1}{4}g_{\mu\nu}\gamma^{\alpha\beta i}f_{i\alpha\beta} \tag{41}$$

We now use the symmetry of $G_{\mu\nu}$ to obtain

$$G_{\mu\nu} = \tfrac{1}{2}(J_{\mu i}h^{i}{}_{\nu} + J_{\nu i}h^{i}{}_{\mu}) - B_{\mu\nu} \tag{42}$$

where

$$B_{\mu\nu} = \tfrac{1}{2}(\gamma^{\alpha}{}_{\mu i}f^{i}{}_{\alpha\nu} + \gamma^{\alpha}{}_{\nu i}f^{i}{}_{\alpha\mu}) - \tfrac{1}{4}g_{\mu\nu}\gamma^{\alpha\beta i}f_{i\alpha\beta} \tag{43}$$

Next, we define

$$M_{\mu\nu} = M^{\alpha}{}_{\mu i} M_{\alpha\nu}{}^{i} - \tfrac{1}{4} g_{\mu\nu} M^{\alpha\beta i} M_{\alpha\beta i} \tag{44}$$

$$P_{\mu\nu} = P^{\alpha}{}_{\mu i} P_{\alpha\nu}{}^{i} - \tfrac{1}{4} g_{\mu\nu} P^{\alpha\beta i} P_{\alpha\beta i}$$

where

$$P_{\mu\nu i} = \tfrac{2}{3}\gamma_{\mu\nu i} - \tfrac{1}{3} f_{i\mu\nu} \tag{45}$$

From equations (29) and (45), it is easily verified that $\frac{9}{8}(M_{\mu\nu} - P_{\mu\nu}) = B_{\mu\nu}$. Thus, we have

$$G_{\mu\nu} = \tfrac{1}{2}(J_{\mu i} h^{i}{}_{\nu} + J_{\nu i} h^{i}{}_{\mu}) + \tfrac{9}{8}(P_{\mu\nu} - M_{\mu\nu}) \tag{46}$$

3. EQUATIONS OF MOTION

We now consider the general equations of motion that are derived from our Einstein equations. Let $\mathcal{T}_{\mu\nu}$ be any tensor of the form

$$\mathcal{T}_{\mu\nu} = F^{\alpha}{}_{\mu} F_{\alpha\nu} - \tfrac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \tag{47}$$

where $F_{\mu\nu}$ is any antisymmetric tensor. A straightforward calculation shows that

$$\mathcal{T}_{\mu\nu}{}^{;\mu} = F^{\mu\alpha}{}_{;\alpha} F_{\mu\nu} + \tilde{F}^{\mu\alpha}{}_{;\alpha} \tilde{F}_{\mu\nu} \tag{48}$$

where $\tilde{F}_{\mu\nu}$ is the dual of $F_{\mu\nu}$, defined in the usual way; see, e.g., Weber (1961). (The conserved current $\tilde{F}_{\mu\nu}{}^{;\nu}$ vanishes if $F_{\mu\nu}$ is the curl of a vector.)

3.1. Equations of Motion (First Form)

By using the fact that $G_{\mu\nu}{}^{;\mu} = 0$, we find from equation (19) that

$$h^{i\mu} j_{i\nu;\mu} + j^{i\mu}\gamma_{i\mu\nu} - (h^{i\mu} j_{i\mu})_{;\nu} - 2\gamma_{\mu\nu}{}^{;\mu} + \tfrac{1}{2}(\gamma^{\alpha\beta i}\gamma_{\alpha\beta i})_{,\nu} \tag{49}$$

Notice that

$$(h^{i\mu} j_{i\mu})_{;\nu} = h^{i\mu}{}_{;\nu} j_{i\mu} + h^{i\mu} j_{i\mu;\nu} = j^{i\mu}\gamma_{i\mu\nu} + h^{i\mu} j_{i\mu;\nu}$$

Upon using this and the relation $f_{i\mu\nu} = \gamma_{i\nu\mu} - \gamma_{i\mu\nu}$ in equation (49), we get

$$h^{i\mu}(j_{i\nu;\mu} - j_{i\mu;\nu}) + j^{i\mu} f_{i\mu\nu} - 2\gamma_{\mu\nu}{}^{;\mu} + \tfrac{1}{2}(\gamma^{\alpha\beta i}\gamma_{\alpha\beta i})_{,\nu} = 0 \tag{50}$$

From equation (48), we easily see that

$$\gamma_{\mu\nu}{}^{;\mu} = \gamma^{\mu\alpha i}{}_{;\alpha}\gamma_{\mu\nu i} + \tilde{\gamma}^{\mu\alpha i}{}_{;\alpha}\tilde{\gamma}_{\mu\nu i} = J^{\mu i}\gamma_{\mu\nu i} + \tilde{J}^{\mu i}\tilde{\gamma}_{\mu\nu i}$$

where $J^{\mu i} = \gamma^{\mu\nu i}{}_{;\nu}$ and $\tilde{J}^{\mu i} = \tilde{\gamma}^{\mu\nu i}{}_{;\nu}$ are the conserved currents that are sources of the fields $\gamma_{\mu\nu i}$ and $\tilde{\gamma}_{\mu\nu i}$, respectively. By using this expression for $\gamma_{\mu\nu}{}^{;\mu}$ in equation (50), we obtain the equations of motion

$$h^{i\mu}\omega_{i\mu\nu} + j^{i\mu}f_{i\mu\nu} - 2J^{\mu i}\gamma_{\mu\nu i} - 2\tilde{J}^{\mu i}\tilde{\gamma}_{\mu\nu i} + \tfrac{1}{2}(\gamma^{\alpha\beta i}\gamma_{\alpha\beta i})_{,\nu} = 0 \tag{51}$$

where $\omega_{i\mu\nu} = j_{i\nu,\mu} - j_{i\mu,\nu}$. The last term in equation (51) is present because of the "cosmological" term in equation (19). All other terms are "Lorentz-force-like." The first acts upon the tetrad itself (the observer frame). The others act upon the currents $j^{i\mu}$, $J^{\mu i}$, and $\tilde{J}^{\mu i}$.

3.2. Equations of Motion (Second Form)

We obtain a simple form for the equations of motion from equation (46). If we multiply equation (46) by $g^{\mu\nu}$, we find that $J_{\alpha i}h^{i\alpha} + R = 0$. Thus, by subtracting $\frac{1}{2}g_{\mu\nu}(J_{\alpha i}h^{i\alpha} + R)$ from the right side of equation (46), we get

$$G_{\mu\nu} = \tfrac{1}{2}(J_{\mu i}h^{i}{}_{\nu} + J_{\nu i}h^{i}{}_{\mu} - g_{\mu\nu}J_{\alpha i}h^{i\alpha}) + \tfrac{9}{8}(P_{\mu\nu} - M_{\mu\nu}) - \tfrac{1}{2}g_{\mu\nu}R \tag{52}$$

By using $G_{\mu\nu}{}^{;\nu} = 0$, we find from equation (52) that

$$h^{i\mu}(J_{\nu i;\mu} - J_{\mu i;\nu}) + J^{\mu i}f_{i\mu\nu} + \tfrac{9}{4}(P_{\mu\nu}{}^{;\mu} - M_{\mu\nu}{}^{;\mu}) - R_{,\nu} = 0 \tag{53}$$

The analysis which leads from equation (52) to equation (53) is analogous to that which led from equation (19) to equation (50). From equation (48), we see that

$$P_{\mu\nu}{}^{;\mu} - M_{\mu\nu}{}^{;\mu} = p^{\mu i}P_{\mu\nu i} + \tilde{p}^{\mu i}\tilde{P}_{\mu\nu i} - m^{\mu i}M_{\mu\nu i} - \tilde{m}^{\mu i}\tilde{M}_{\mu\nu i} \tag{54}$$

where $p^{\mu i} = P^{\mu\nu i}{}_{;\nu}$, $\tilde{p}^{\mu i} = \tilde{P}^{\mu\nu i}{}_{;\nu}$, $m^{\mu i} = M^{\mu\nu i}{}_{;\nu}$, and $\tilde{m}^{\mu i} = \tilde{M}^{\mu\nu i}{}_{;\nu}$. Now, it is clear from equations (29) and (45) that $M_{\mu\nu i} - P_{\mu\nu i} = \frac{2}{3}f_{i\mu\nu}$. But, $\tilde{f}_{i\mu\nu}{}^{;\nu}$ vanishes because $f_{i\mu\nu}$ is the curl of a vector. Thus, $\tilde{M}^{\mu\nu i}{}_{;\nu} = \tilde{P}^{\mu\nu i}{}_{;\nu} = \frac{2}{3}\tilde{\gamma}^{\mu\nu i}{}_{;\nu}$. In this way, we see that $\tilde{m}^{\mu i} = \tilde{p}^{\mu i} = \frac{2}{3}\tilde{J}^{\mu i}$, so that equation (53) may be written

$$h^{i\mu}(J_{\nu i;\mu} - J_{\mu i;\nu}) + J^{\mu i}f_{i\mu\nu} + \tfrac{9}{4}(p^{\mu i}P_{\mu\nu i} - m^{\mu i}M_{\mu\nu i}) - \tilde{J}^{\mu i}\tilde{f}_{i\mu\nu} - R_{,\nu} = 0 \tag{55}$$

From equations (29) and (45),

$$p^{\mu i}P_{\mu\nu i} - m^{\mu i}M_{\mu\nu i} = -\tfrac{4}{9}(J^{\mu i}f_{i\mu\nu} + j^{i\mu}\gamma_{\mu\nu i})$$

so equation (55) becomes

$$h^{i\mu}\Omega_{\mu\nu i} - j^{i\mu}\gamma_{\mu\nu i} - \tilde{J}^{\mu i}\tilde{f}_{i\mu\nu} - R_{,\nu} = 0 \tag{56}$$

where $\Omega_{\mu\nu i} = J_{\nu i,\mu} - J_{\mu i,\nu}$. As in equation (51), the last term in equation (56) is cosmological, and all others are Lorentz-force-like.

4. AN ALTERNATIVE THEORY

We previously (Pandres, 1981, 1984a,b) considered the variational principle

$$\delta \int C^{\mu} C_{\mu} \sqrt{-g}\, d^4 x = 0 \tag{57}$$

where $h^i{}_\mu$ is varied. This yields field equations which just state that the quantity $C_i = C_\mu h_i{}^\mu$ is constant and lightlike. These field equations, in turn, imply the validity of Einstein equations that are identical in form to those which are implied by $C_\mu = 0$. The equations of motion that are derived from these Einstein equations differ from equations (51) and (56) through the presence of a term which vanishes if $C_\mu = 0$.

ACKNOWLEDGMENT

It is a pleasure to thank Prof. E. L. Green for stimulating discussions and helpful remarks.

REFERENCES

Bade, W. L., and H. Jehle (1953). *Reviews of Modern Physics,* **25**, 714–728.
Crease, R. P., and C. C. Mann (1986). *The Second Creation,* Macmillan, New York, p. 277.
Dirac, P. A. M. (1978). *Directions in Physics,* Wiley, New York, p. 41.
Eddington, A. E. (1957). *The Mathematical Theory of Relativity,* 2nd ed., Cambridge University Press, Cambridge, p. 222.
Eisenhart, L. P. (1925). *Riemannian Geometry,* Princeton University Press, Princeton, New Jersey, p. 97.
Loos, H. G. (1963). *Annals of Physics,* **25**, 91–108.
Moriyasu, K. (1983). *An Elementary Primer for Gauge Theory,* World Scientific, Singapore, p. 110.
Pandres, D., Jr. (1962). *Journal of Mathematical Physics,* **3**, 602–607.
Pandres, D., Jr. (1981). *Physical Review D,* **24**, 1499–1508.
Pandres, D., Jr. (1984a). *Physical Review D,* **30**, 317–324.
Pandres, D., Jr. (1984b). *International Journal of Theoretical Physics,* **23**, 839–842.
Pandres, D., Jr. (1995). *International Journal of Theoretical Physics,* **34**, 733–759.
Salam, A. (1968). In *Proceedings of the 8th Nobel Symposium on Elementary Particle Theory,* N. Svartholm, ed., Almquist Forlag, Stockholm, pp. 367–377.
Schrödinger, E. (1960). *Space-Time Structure,* Cambridge University Press, Cambridge, pp. 97, 99.
Synge, J. L. (1960). *Relativity: The General Theory,* North-Holland, Amsterdam, pp. 14, 357.
Weber, J. (1961). *General Relativity and Gravitational Waves,* Interscience, New York, p. 147.
Weinberg, S. (1967). *Physical Review Letters,* **19**, 1264–1266.
Witten, E. (1988). *Superstrings: A Theory of Everything?,* P. C. W. Davis and J. Brown, eds., Cambridge University Press, Cambridge, p. 90.
Yang, C. N. (1974). *Physical Review Letters,* **33**, 445, and references cited therein.